Service Overlay Network Design with Reliability Constraints

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***Abstract*—We studied a class of Service Overlay Network (SON) design problem with reliability constraints. It is assumed that a SON network could enter an inadmissible status for two reasons; first when there is insufficient resource to accommodate new connections, second when some hardware devices malfunction. The design problem is usually formulated as either a Maximum Profit (MP) constrained optimization problem or a Minimum Cost (MC) constrained optimization problem. In this article we investigate the relationship between the two formulations in the context of ensuring system operability. By using the set of Lagrange multipliers from the MC formulation as a tool, we show the general condition that MP and MC give exactly the same network designs. The key contribution of this paper is the provision of insight into the solution nature of the MP and the MC formulations in designing a reliable overlay network, thereby giving guidelines to the proper formulation the network designers may consider in designing a reliable yet economically optimal SON network.**

Keywords- Reliability; Service Guarantees; Network Design; Profit Maximization; Cost Minimization

1 INTRODUCTION

The demand for end-to-end Quality of Service (QoS) guarantees in the Internet has increased significantly due to the introduction of new applications like VoIP, online gaming, and video conferencing. This poses a major challenge to the current Internet architecture. Owing to historical reasons, the Internet consists of a large collection of independent Autonomous Systems (ASes). To ensure end-to-end QoS guarantees for the data, one has to build a multi-lateral business relationship with all the independent ASes his data transit. This makes it unrealistic to obtain end-to-end QoS guarantees. A higher level mechanism on the top of the Internet known as Service Overlay Network (SON) is thus proposed to alleviate this problem [11]. The SON network operates in a manner similar to a virtual network. The SON operator owns the SON gateways which are placed in strategic locations. To realize the SON network, the SON operator leases bandwidths with QoS guarantees from the underlying Autonomous Systems (ASes), in the form of Service Level Agreements (SLAs). The leased bandwidths act as logical links that connect the SON gateways. Each logical link consists of exactly one autonomous system. Once all the logical links are in place, the SON is realized and the overlay network formed is under the administration of a single authority. Because the SON is administrated by a single operator, it is capable of providing end-to-end QoS guarantees to the value-added services provided by it (i.e. online gaming, Video conferencing, etc). A user with access to the Internet can access the service gateway to use the value-added services, provided the hosts holding the contents are also connected to some SON service gateways. In a SON, the connections are classified by the origin and the destination (OD) gateways. Users pay the service charge based on the origins and

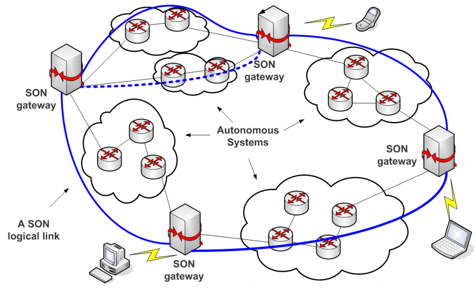

Fig. 1. An example SON network.

destinations of their connections as well as their connection durations. Figure 1 shows an example of the SON network.

It is quite common that in formulating network design problems, one may be confronted with the choice of choosing to maximize the profit or to minimize the cost, as the design criteria. An interesting question is thus raised: since the maximum profit (MP) and the minimum cost (MC) approaches optimize different objectives, how do their solutions compare and how does a network designer decide which approach to employ. These questions could be non-trivial under the SON environment. This is because the SON operators enjoy great degree of freedom in deploying their own inter-domain routing schemes as the network is solely under their administrations. Owning to this flexibility, different network administrators are likely to employ different routing schemes to fit their business objectives. The routing scheme employed would influence the network design significantly. The conclusions drawn regarding SON design for a particular routing scheme may not be valid for another. This paper addresses this objective choice problem faced by the operators for a class of Service Overlay Network (SON) capacity allocation problems. The result does not depend on a specific routing scheme as long as certain conditions are met. This paper is structured as follows: Section 2 is the description of the problem assumptions and formulations, Section 3 discusses the major results, Section 4 shows a simple example that illustrates the results, Section 5 is the conclusion section that discusses the conclusions obtained and possible future extensions.

2 THE PROBLEM

2.1 Problem Formulations

To deploy a SON, a major challenge faced by the operator would be the optimal amount of bandwidths to be leased on the logical links. The allocated bandwidths should fit the economic objective of the operator yet meet the user expectation regarding the Grade of Service (GoS) levels. The Grade of Service (GoS) levels are usually quantified by connection blocking probabilities. The connection blocking includes the blockings due to resource constraints and also the blockings resulting from hardware failures. This blocking probability is viewed as a reliability measure of the network by the users. By considering the SON as a loss network [5], the design problem can be formulated as a constrained maximum profit problem (MP) as listed in (2.1.1a). The problem is assumed to be solved using the Lagrangian relaxation method [3].

$$\max_{N_s} \quad \sum_{ij} \lambda^{ij} w^{ij} (1 - B^{ij}) - \sum_s C_s(N_s)$$

$$B^{ij} \le \overline{L}^{ij} \qquad u_{ij}$$

$$N_s \ge 0 \qquad z_s \qquad (2.1.1a)$$

$$c_s = \sum_{ij} (\lambda^{ij} w^{ij} + u_{ij}^*)(\frac{-\partial B^{ij}}{\partial N_s^*}) \quad \forall \text{s, s.t. } N_s^* > 0 \qquad (2.1.1b)$$

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix}$$

$$h_{nm} = \sum_{ij} (w^{ij} \lambda^{ij} + u_{ij}^*) \frac{\partial^2 B^{ij}}{\partial N_n \partial N_m} \quad , \; h_{nn} = \sum_{ij} (w^{ij} \lambda^{ij} + u_{ij}^*) \frac{\partial^2 B^{ij}}{\partial N_n^2} \qquad (2.1.1c)$$

The corresponding first order optimality condition and the Hessian matrix of the relaxed problem are listed in (2.1.1b) and (2.1.1c) respectively. We shall denote formulation (2.1.1a) as MP in the remainder of this article. In the formulation (2.1.1a), λ^{ij} is the given poissonian connection arrival rate demanding the connection of the node pair *(i,j)* (i.e. origin gateway is *i*, destination gateway is *j*), w^{ij} is the expected service charge/expected reward generated by an admitted *(i,j)* connection. We shall use the terms "reward" and "service charge" interchangeably in this article. The notion B^{ij} denotes the analytical end-to-end blocking probability for connections of the node pair *(i,j)*, and it is routing scheme dependent. A connection can be blocked for three reasons: a) there is no available resource, b) hardware failures on the logical links, c) a combination of (a) and (b). The GoS constraint for each OD pair *(i,j)* is given by $\overline{L}^{ij}$, which specifies the maximum allowed end-to-end blocking probability due to resource constraints or hardware failures. The capacity of a link *s* is denoted by N_s and it is a decision variable of this problem. The function $C_s(.)$ is the cost function that quantifies the cost of allocating N_s units of capacities on link *s* per unit time, based on some SLA. This function is assumed to be a linear function of the variable N_s. The variables u_{ij} and z_s are the Lagrange multipliers with respect to the constraints. The minimum cost formulation (MC) of the same problem, the corresponding first order optimality condition and the Hessian matrix are given by (2.1.2a), (2.1.2b) and (2.1.2c) respectively. The only new notation introduced in (2.1.2a) is v_{ij}. It is the Lagrange multiplier corresponds to the GoS constraint. We shall denote formulation (2.1.2a) as MC in the rest of the article. It is assumed that both MP and MC formulations employ the same (optimal) routing scheme in the routing layer.

$$\min_{N_s} \quad \sum_s C_s(N_s)$$

$$B^{ij} \le \overline{L}^{ij} \qquad v_{ij}$$

$$N_s \ge 0 \qquad z_s \qquad (2.1.2a)$$

$$c_s = \sum_{ij} v_{ij}^* (\frac{-\partial B^{ij}}{\partial N_s^*}) \qquad \forall \text{s, s.t. } N_s^* > 0 \qquad (2.1.2b)$$

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix}, \; h_{nm} = \sum_{ij} v_{ij}^* \frac{\partial^2 B^{ij}}{\partial N_n \partial N_m} \quad , \; h_{nn} = \sum_{ij} v_{ij}^* \frac{\partial^2 B^{ij}}{\partial N_n^2} \quad (2.1.2c)$$

To analyze the two different formulations, a typical approach is to investigate the corresponding optimality conditions since they provide the richest information regarding the optimal solutions. From (2.1.1b) and (2.1.2b), one can see that provided B^{ij} are the same, the first order necessary conditions of the two formulations are equivalent if the expressions v_{ij}^* and $\lambda^{ij} w^{ij} + u_{ij}^*$ are equal for all the OD pairs (*i,j*). In this case the second order sufficient conditions of the MP and MC will also be the same, as the Hessian matrices (2.1.1c) and (2.1.2c) are identical then. Therefore, regardless of their different objectives, MP and MC *could* deliver the same solutions if one can set the multipliers u_{ij}^* appropriately, subject to the dual feasibility conditions. As we shall see, there exists a general condition that the MP and MC formulations can be made equivalent. When the solutions of MP and MC coincide, it indicates the cost function dominates the reward function so that the incorporation of additional reward information does not differentiate the solutions of MP from the solutions of MC. We shall show that for some specific range of reward parameters w^{ij}, the costs dominate the design. While in other range of the w^{ij} parameters, the reward parameters are large enough to influence the network design.

2.2 The End-to-End Blocking Probabilities

Before the analysis can be made, a proper analytical form of the blocking function, B^{ij}, is needed. The functional form of B^{ij} varies with routing schemes [4]. We employ techniques from the reliability theory [9] to devise a general and functional form for B^{ij}, regardless of the actual routing scheme employed. The B^{ij} function derived below is primarily based on the reduced load approximation model [10], which assumed statistical link independence and Poisson link arrival rates.

We consider the *collection* of network paths, that connect a particular origin node i with a particular destination node j, as a complete system. The task of this system is to serve the connections between the node pair (i,j). Assume the network links are independent of one another. The links in the *collection* of paths are the independent components of the system. Denote these links by s and let R_{ij} be a set that contains all such links. Define an indicator variable y_s for the link s. Whereas y_s equals to *zero* if link s is functioning and has enough resource to admit at least one connection, and y_s equals to *one* if link s has failed or does not have resource to serve a new connection. The expected value of y_s is the blocking probability of link s due to resource constraints or hardware problems. According to the reliability theory [9], a Boolean function $\bar{\phi}(Y)$ that indicates whether the system is unavailable for a new (i,j) connection can be defined by taking the link status vector $Y=[y_s]$ as the input. The expected value of $\bar{\phi}(Y)$ is effectively B^{ij}, the end-to-end blocking probability for connection pair (i,j). Since y_s are independent zero-one random variables and $\bar{\phi}(Y)$ is a Boolean function, we can perform the Shannon decomposition on the function $\bar{\phi}(Y)$. Using link s as the pivot we have (2.2.1):

$$\begin{aligned}\bar{\phi}(Y) &= y_s\bar{\phi}(1_s,Y) + (1-y_s)\bar{\phi}(0_s,Y)\\ &= \bar{\phi}(0_s,Y) + [\bar{\phi}(1_s,Y) - \bar{\phi}(0_s,Y)]y_s \qquad (2.2.1)\end{aligned}$$

$(0_s,Y)$ and $(1_s,Y)$ are the status vectors that differs only in the s^{th} link. The functions $\bar{\phi}(0_s,Y)$ and $\bar{\phi}(1_s,Y)$ indicates that whether the system has been blocked given that the link s is in admissible status/ blocking status. These two functions do not depend on the link s. Since the links are independent, taking expectation on the both sides of (2.2.1) yields (2.2.2).

$$E[\bar{\phi}(Y)] = f^{ij}_{1,s}(E[y_{s'}]) + f^{ij}_{2,s}(E[y_{s'}])E[y_s] \qquad (2.2.2)$$

Where $s' \neq s$, are the links in the set R_{ij}. In the SON architecture assumed [11], each pair of SON gateways are separated by one autonomous system, so a "link" always consists of one-hop. By the definition of y_s, the expectation $E[y_s]$ is the probability for a single link to be in blocking status. We assume link failures due to hardware problems to be independent of the resource related blockings. We also assume the link connection arrival rates to be independent Poisson processes. Hence we have $E[y_s]= E_s(\lambda_s,N_s) + h_s - E_s(\lambda_s,N_s)h_s$. Where λ_s is link connection arrival rate at equilibrium, where $E_s(\lambda_s,N_s)$ is the Erlang-B formula, and h_s is a constant to denote the probability for link s to fail owning to hardware issues (i.e. random hardware failures, loss of electricity, etc), this probability can be estimated from the historical data of the AS. Therefore (2.2.2) can be written to (2.2.3).

$$\begin{aligned}B^{ij} = {} & f^{ij}_{1,s}(\lambda_{s'},N_{s'},h_{s'}) + f^{ij}_{2,s}(\lambda_{s'},N_{s'},h_{s'})h_s\\ & + f^{ij}_{2,s}(\lambda_{s'},N_{s'},h_{s'})(1-h_s)E_s(\lambda_s,N_s) \qquad (2.2.3)\end{aligned}$$

It should be clear that $B^{ij}=E[\bar{\phi}(Y)]$ is essentially a reduced-load approximation of the OD pair blocking probability, as link independence and Poisson link arrival rates are assumed. Note that in (2.2.3), $f^{ij}_{1,s} \in [0,1]$, as it is the blocking probability of the system given that link s is in admissible state (refer to 2.2.1). We also need $f^{ij}_{2,s} \geq 0$ if the end-to-end blocking probability B^{ij} is non-increasing in the presence of additional available link capacities (when $f^{ij}_{2,s} \geq 0$, it is the Birnbaum's importance measure of link s). This is a monotonic property we imposed on the routing schemes and it should be satisfied by any reasonable (centralized) routing schemes since such schemes should not perform poorer with more available resources. We also impose the assumption that the routing objective function being uni-modal with respect to the capacity assignment. Moreover, we assume the continuous extension of Erlang-B formula suggested in [1] is used throughout this article.

Expression (2.2.3) is valid for any link s. So we can represent B^{ij} in the form of $f^{ij}_{1,s} + f^{ij}_{2,s}h_s + f^{ij}_{2,s}(1-h_s)E_s(\lambda_s,N_s)$ for any link s, where $f^{ij}_{1,s}$, $f^{ij}_{2,s}$ are independent of link s. This general expression of B^{ij} facilitates easy access to the derivatives of B^{ij} with respect to the capacity on an arbitrary link. In a similar spirit, by applying the Shannon decomposition again on expression (2.2.3), B^{ij} can be further decomposed in terms of the blocking probabilities of link s and link s', as in (2.2.4).

$$B^{ij} = g_0 + g_1E_s(.) + g_2E_{s'}(.) + g_3E_s(.)E_{s'}(.)$$

$$\begin{aligned}g_0 &= f^{ij}_{1,s}(0_{s'}) + f^{ij}_{2,s}(0_{s'})h_s + (f^{ij}_{1,s}(1_{s'}) - f^{ij}_{1,s}(0_{s'}))h_{s'}\\ &\quad + (f^{ij}_{2,s}(1_{s'}) - f^{ij}_{2,s}(0_{s'}))h_sh_{s'}\\ g_1 &= (1-h_s)(f^{ij}_{2,s}(0_{s'}) + (f^{ij}_{2,s}(1_{s'}) - f^{ij}_{2,s}(0_{s'}))h_{s'})\\ g_2 &= (1-h_{s'})((f^{ij}_{1,s}(1_{s'}) - f^{ij}_{1,s}(0_{s'})) + (f^{ij}_{2,s}(1_{s'}) - f^{ij}_{2,s}(0_{s'}))h_s)\\ g_3 &= (1-h_s)(1-h_{s'})(f^{ij}_{2,s}(1_{s'}) - f^{ij}_{2,s}(0_{s'})) \qquad (2.2.4)\end{aligned}$$

The notions $0_{s'}$ and $1_{s'}$ in (2.2.4) indicate the status of the link s'. For instance, $f^{ij}_{1,s}(1_{s'})$ indicates the function $f^{ij}_{1,s}(.)$ given that link s' has been blocked. Expression (2.2.4) can be employed to derive the second order information of the MP and MC formulations. The functions g_0, g_1, g_2 and g_3 in (2.2.4) are independent of the links s and s'.

By using expressions (2.2.3), the optimality conditions (2.1.1b) and (2.1.2b) can be rewritten to (2.2.5) and (2.2.6).

$$[\sum_{ij}(\lambda^{ij}w^{ij}+u_{ij}^*)\times f_{2,s}^{ij}(\lambda_{s'},N_{s'})](1-h_s)(\frac{-\partial E_s(\lambda_s,N_s)}{\partial N_s^*})$$

$$\forall \text{s, s.t. } N_s^*>0 \qquad (2.2.5)$$

$$[\sum_{ij}v_{ij}^*\times f_{2,s}^{ij}(\lambda_{s'},N_{s'})](1-h_s)(\frac{-\partial E_s(\lambda_s,N_s)}{\partial N_s^*})$$

$$\forall \text{s, s.t. } N_s^*>0 \qquad (2.2.6)$$

Assume the continuous Erlang-B formula in [1] is employed. Since it is a C^{∞} function [2], so the second derivatives exist. By using (2.2.3) and (2.2.4) the elements of the Hessian matrices of (2.1.1c) and (2.1.2c) can be re-written to (2.2.7) and (2.2.8) respectively.

$$h_{nm}=[\sum_{ij}(w^{ij}\lambda^{ij}+u_{ij}^*)(f_{2,n}^{ij}(1_m)\text{-}f_{2,n}^{ij}(0_m))](1-h_n)(1-h_m)\frac{\partial E_n(.)}{\partial N_n}\ \frac{\partial E_m(.)}{\partial N_m}$$

$$h_{nn}=[\sum_{ij}(w^{ij}\lambda^{ij}+u_{ij}^*)f_{2,n}^{ij}(.)](1-h_n)\frac{\partial^2 E_n(.)}{\partial N_n^2} \qquad (2.2.7)$$

$$h_{nm}=[\sum_{ij}v_{ij}^*(f_{2,n}^{ij}(1_m)\text{-}f_{2,n}^{ij}(0_m))](1-h_n)(1-h_m)\frac{\partial E_n(.)}{\partial N_n}\ \frac{\partial E_m(.)}{\partial N_m}$$

$$h_{nn}=[\sum_{ij}v_{ij}^*f_{2,n}^{ij}(.)](1-h_n)\frac{\partial^2 E_n(.)}{\partial N_n^2} \qquad (2.2.8)$$

3 THE MP AND MC FORMULATION

3.1 The General Conditions for the Equivalence of MP and MC Formulations

Note that from the new optimal conditions (2.2.5) and (2.2.6), if the sums $\sum_{ij\in\theta_s}(\lambda^{ij}w^{ij}+u_{ij}^*)\times f_{2,s}^{ij}(.)$ and $\sum_{ij\in\theta_s}v_{ij}^*\times f_{2,s}^{ij}(.)$ can be made identical for all the links at the optimal solution of MC, the first order optimality conditions of MP and MC will be identical. The second order optimality conditions will then also be identical. To see this, we substitute (2.2.7) and (2.2.8) into the Hessian matrices of (2.1.1c) and (2.1.2c). Recall that $f_{2,s}^{ij}(0_{s'})$ and $f_{2,s}^{ij}(1_{s'})$ are the function $f_{2,s}^{ij}$ given that link s' is in admissible status/has been blocked. So $\sum_{ij\in\theta_s}(\lambda^{ij}w^{ij}+u_{ij}^*)\times f_{2,s}^{ij}(.)=\sum_{ij\in\theta_s}v_{ij}^*\times f_{2,s}^{ij}(.)$ implies $\sum_{ij\in\theta_s}(\lambda^{ij}w^{ij}+u_{ij}^*)\times f_{2,s}^{ij}(y_{s'})=\sum_{ij\in\theta_s}v_{ij}^*\times f_{2,s}^{ij}(y_{s'})$ where $y_{s'}$ is either *0* or *1*. It is then easy to see the Hessian matrices of the two formulations are identical. Hence, the problem is now reduced to showing the existence of a set of non-negative (dual feasible) multipliers u_{ij}^*, such that the two aforesaid sums can be made equal. We shall show the general conditions for the multipliers to exist using Farkas's lemma [8]. No assumption regarding the number of OD pairs and network links is made. Without loss of generality, let's assume the multipliers v_{ij}^* exist and the OD pairs are indexed from 1 to $|ij|$, where $|ij|$ is number of distinct OD pairs. Let's also assume the links are indexed from 1 to $|s|$, where $|s|$ is the total number of links in the network. Define a matrix ***A*** and a vector ***B*** as in (3.1.1). Then the non-negative matrix ***A*** has a dimension of $|s|\times|ij|$. The function $f_{2,s}^{ij=k}(.)$ in the matrix ***A*** indicates it is a function corresponds to the k^{th} OD pair and it is the f_2^{ij} function of the link s. This function takes the optimal solution of the MC formulation as the input in (3.1.1) and (3.1.2). The vector ***B*** has a dimension of $|s|\times 1$. The problem of finding a set of non-negative u_{ij}^* is equivalent to finding a non-negative vector ***U**** such that ***AU*=B***.

$$A=\begin{bmatrix} f_{2,1}^{ij=1}(.) & \cdots & f_{2,1}^{ij=|ij|}(.) \\ \vdots & \ddots & \vdots \\ f_{2,|s|}^{ij=1}(.) & \cdots & f_{2,|s|}^{ij=|ij|}(.) \end{bmatrix} \quad B=\begin{bmatrix} \sum_{ij\in\theta_1}(v_{ij}^*-\lambda^{ij}w^{ij})\times f_{2,1}^{ij}(.) \\ \vdots \\ \sum_{ij\in\theta_s}(v_{ij}^*-\lambda^{ij}w^{ij})\times f_{2,|s|}^{ij}(.) \end{bmatrix} \qquad (3.1.1)$$

$$A^Tp=\begin{bmatrix} \sum_s p_s f_{2,s}^{ij=1}(.) & ,\cdots, & \sum_s p_s f_{2,s}^{ij=|ij|}(.) \end{bmatrix}^T=[\eta^1,...,\eta^{|ij|}]^T\ \geq 0^T \qquad (3.1.2)$$

By the Farkas's lemma, the non-negative vector ***U**** exists, if and only if for every vector ***P*** such that $\boldsymbol{A^TP\geq 0}$, the inner product $\boldsymbol{B^TP}$ is non-negative. That means for any vector ***P*** that satisfies (3.1.2), the expression (3.1.3) must be positive.

$$\begin{aligned} <B,p> &= \sum_s p_s\sum_{ij}(v_{ij}^*-\lambda^{ij}w^{ij})\times f_{2,s}^{ij}(.) \\ &= \sum_{ij}(v_{ij}^*-\lambda^{ij}w^{ij})\sum_s p_s\times f_{2,s}^{ij}(.) \\ &= \sum_{ij}d^{ij}\eta^{ij} \end{aligned} \qquad (3.1.3)$$

In (3.1.3), $d^{ij}=(v_{ij}^*\text{-}\lambda^{ij}w^{ij})$ are not restricted in sign. The symbols η^{ij} denotes the elements of (3.1.2) which are always non-negative. From the expressions (3.1.1) and (3.1.2), we note that it is possible to set some η^{ij} arbitrarily large by properly picking value(s) for p_s for some instances of ***A***. Since d^{ij} are not restricted in sign, the inner product of (3.1.3) can become negative unless some further constraints are imposed on the routing matrix ***A***. The only case that (3.1.3) is always positive, regardless of the ***A*** matrix, occurs when all the d^{ij} are non-negative. This implies condition (3.1.4). The result can be alternatively verified by the regularity conditions.

$$0\leq w^{ij}\lambda^{ij}\leq v_{ij}^* \qquad \forall ij \qquad (3.1.4)$$

Condition (3.1.4) is the sufficient condition such that the MP and MC formulations can be made equivalent, regardless of the routing scheme employed. So the following results are valid for all routing schemes that are monotonic and uni-modal. We first need to establish the uniqueness of the allocated link capacity with respect to the link cost, a set of real values v_{ij}, and the link connection intensity vector Λ_s (i.e. offered connection intensities which is equal to $A\Lambda$, where $\Lambda=[\lambda_{ij}]$). Define θ_s to be a set which contains indexes of all the

OD pairs that utilize link s in at least one of their paths. Let N_s be the capacity of a link s, which is assumed to be continuous. Let c_s be the cost of allocating one unit of capacity on link s, and the constant h_s to be the hardware failure probability on link s. Let $E_s(\lambda_s, N_s)$ denotes the Erlang-B formula with offered connection intensity λ_s and link capacity of N_s as parameters.

Lemma 1
For positive constants v_{ij}, c_s, and fixed Λ_s, there exists at most one N_s on the link s that satisfies the following equation:

$$c_s = [\sum_{ij\in\theta_s} v_{ij} \times f_{2,s}^{ij}(\lambda_{s'}, N_{s'})](1-h_s)(\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s}) \qquad (3.1.5)$$

Proof:
Assume the continuous extension of Erlang-B formula suggested in [1] is being employed. Note that the function f_2^{ij}= $E[\bar{\phi}(1,Y)-\bar{\phi}(0,Y)]$ is the same as the one defined earlier and it is positive (because we imposed the condition $ij\in\theta_s$ to ensure it will not be zero). The values λ_s are the elements of Λ_s. We rearrange the terms and get (3.1.6):

$$\frac{c_s}{[\sum_{ij\in\theta_s} v_{ij} \times f_{2,s}^{ij}(\lambda_{s'}, N_{s'})](1-h_s)} = (\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s}) \qquad (3.1.6)$$

Since the functions $f_{2,s}^{ij}$ is independent of N_s. It can be regarded as a positive constant functions with respect to N_s. Denote $(\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s})$ by $f_3(\lambda_s, N_s)$. It is known that the continuous Erlang-B formula is a C^∞ function [2], which is strictly convex in the capacity [1]. Therefore for a fixed λ_s, the function $f_3(.)$ is strictly decreasing and continuous in N_s, and there is an one-to-one correspondence between the function's value and N_s for the fixed λ_s. Moreover it can be seen that $f_3(.)$ is positive , since when N_s is increased by delta, the function $-E_s(.)$ always increases if λ_s is fixed. So for positive constants v_{ij}, c_s, and fixed λ_s, there is a unique N_s that satisfies equation (3.1.6). There are two special cases for expression (3.1.6), first if the LHS of (3.1.6) is larger than $\max_{N_s}(f_3(\lambda_s, N_s))$ then N_s does not exist, second if the cost c_s equals to zero, then N_s equals to infinite. □

The physical interpretation of Lemma 1 is that when the link connection intensity is fixed, then for each set of c_s and v_{ij}, there is at most one N_s value that solves (3.1.5) and thereby satisfying the first order necessary conditions. Lemma 1 provides a tool for us to show the conditions such that MC and MP are equivalent. The following theorem will establish the relationship between MC and MP under some specified conditions.

Theorem 1
Consider a MP formulation (2.1.1a) and a MC formulation (2.1.2a) of the SON capacity allocation problem. If a set of Lagrange multipliers v_{ij}^* exists for the MC formulation, and if the condition (3.1.4) is satisfied, then an optimal solution of the MC formulation is also an optimal solution of the MP formulation. Further to this, if the set of v_{ij}^* is unique, then MP and MC formulations are equivalent.

Proof:
The stationary conditions for MP and MC are (3.1.7) and (3.1.8) respectively. All the symbols have the same definitions as being defined earlier. Note that (3.1.7) and (3.1.8) both represent n sets of equations where n equals to the number of links in the network.

$$c_s = [\sum_{ij\in\theta_s} (\lambda^{ij} w^{ij} + u_{ij}) \times f_{2,s}^{ij}(\lambda_{s'}, N_{s'})](1-h_s)(\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s}) \qquad (3.1.7)$$

$$c_s = [\sum_{ij\in\theta_s} v_{ij} \times f_{2,s}^{ij}(\lambda_{s'}, N_{s'})](1-h_s)(\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s}) \qquad (3.1.8)$$

Suppose v_{ij}^* are the Lagrange multipliers at optimality for MC. If (3.1.4) is satisfied, then it is possible to find a non-negative vector $\boldsymbol{U^*}$=[u_{ij}^*] such that $\sum_{ij\in\theta_s} (\lambda^{ij} w^{ij} + u_{ij}^*) \times f_{2,s}^{ij}(.)$ equal to $\sum_{ij\in\theta_s} v_{ij}^* \times f_{2,s}^{ij}(.)$ at the optimality of the MC formulation. Assume a vector of link intensities Λ_s^{c*} is designated by some optimal routing rules at the optimal solution of the MC formulation. Suppose the tuple $(\Lambda_s^{c*}, N^{c*}(\Lambda_s^{c*}))$, where $N^{c*}(\Lambda_s^{c*})$ is the vector of optimal capacities, is a solution that satisfies (3.1.8) and also the complementary slackness conditions. Then it is obvious from Lemma 1 that the solution $(\Lambda_s^{c*}, N^{c*}(\Lambda_s^{c*}))$ is unique with respect to Λ_s^{c*}. Now if we substitute the solution $(\Lambda_s^{c*}, N^{c*}(\Lambda_s^{c*}))$ along with $\boldsymbol{U^*}$ into (3.1.7), the solution should satisfy (3.1.7) if it satisfies (3.1.8). This is because $\sum_{ij\in\theta_s} (\lambda^{ij} w^{ij} + u_{ij}^*) \times f_{2,s}^{ij}(.)$ equals to $\sum_{ij\in\theta_s} v_{ij}^* \times f_{2,s}^{ij}(.)$ for all the n equations, and also because the expression $(\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s})$ depends only on Λ_s^{c*} and $N^{c*}(\Lambda_s^{c*})$. The complementary slackness conditions are also satisfied since the constraints of MP and MC are identical. In other words, when (3.1.4) is satisfied, the solution that satisfies the first order necessary condition of MC also satisfies the first order necessary condition of the MP. With the same vector $\boldsymbol{U^*}$, the second order sufficient condition check is trivial, because the Hessian matrices of the corresponding Lagrangian duals are identical (see (2.2.7) and (2.2.8)). Therefore we have shown that the optimal solution of MC is also an optimal solution of MP. Further to this, if the vector $\boldsymbol{U^*}$ is unique and condition (3.1.4) is satisfied, then formulations MP and MC are equivalent. This can be proved by contradiction. If there exists another set of multipliers $X^* = [x_{ij}^*]$ of MP, such that $\sum_{ij\in\theta_s} (\lambda^{ij} w^{ij} + x_{ij}^*) \times f_{2,s}^{ij}(.)) = \sum_{ij\in\theta_s} y_{ij}^* \times f_{2,s}^{ij}(.) \neq \sum_{ij\in\theta_s} v_{ij}^* \times f_{2,s}^{ij}(.)$, where

$x_{ij}^* \geq 0$ and $y_{ij}^* \geq 0$. Suppose for any optimal solution $(\Lambda_s^{p*}, N_s^{p*}(\Lambda_s^{p*}))$ that corresponds to ***X**** and satisfies both the necessary and sufficient conditions of the MP formulation. It is obvious that the multipliers $y_{ij}^* = (\lambda^{ij} w^{ij} + x_{ij}^*)$ along with $(\Lambda_s^{p*}, N^{p*}(\Lambda_s^{p*}))$ satisfy (3.1.8). The complementary slackness conditions are also satisfied because (2.1.1a) and (2.1.2a) have the same constraints. Moreover this solution also satisfies the second order necessarily conditions of MC as long as it satisfies these conditions for MP. That implies ***Y****=[y_{ij}^*] is also a set of multipliers for MC. But this contradicts with the claim that MC has a unique set of multipliers. Therefore it is impossible for the vector ***X**** to exist. As a result MP must also have a unique set of multipliers (i.e. ***U****=[u_{ij}^*]). This shows that both MP and MC give the same optimal solution if (3.1.4) is satisfied, and are therefore equivalent. □

Assume there exists a unique set of multipliers for the MC formulation and the corresponding Λ_s^* is unique; the above theorem indicates that if the service charges are lower than some values (i.e. (3.1.4) is satisfied), then MP and MC give the same solution. This is the region such that the cost function dominates the reward function. For all the other regions, the service charges are large enough to influence the designs and MC and MP are different. Owing to the space limitation we shall discuss the results for one such region in Section 3.2.

3.2 The MP and MC Formulations as Differentiated by the Service Charges

If condition (3.2.1) holds for the service charges w^{ij}, then it is obvious that condition (3.1.4) can not be satisfied, as condition (3.2.1) implies the ***B*** vector to be strongly negative. We shall show the MP formulation is different from the MC formulation in this region in two ways. First, MP provides solutions with *strictly* better GoS guarantees than that of MC. Second, MP also provides solution with *strictly* more profit than the MC formulation. The investigation will provide us with insight into the relationship between the service charges and GoS guarantees (which will be discussed in another paper).

$$0 \leq \sum_{ij \in \theta_s} v_{ij}^* f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*}) < \sum_{ij \in \theta_s} w^{ij} \lambda^{ij} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*}) \quad \forall s \tag{3.2.1}$$

Before we present the main results, again we first need a lemma. Lemma 2 establishes the relation between the optimal capacities allocated and the magnitude of service charges.

Lemma 2

For any two real numbers v^1, v^2, where $v^1 > v^2 > 0$. If λ_s is a constant in (3.2.2), then we have $N_s^{v^1} > N_s^{v^2}$, where $N_s^{v^1}$ and $N_s^{v^2}$ are the values of N_s^v correspond to $v=v^1$ and $v=v^2$ respectively in (3.2.2).

$$c_s = v(-\frac{\partial E_s(\lambda_s, N_s^v)}{\partial N_s^v}) \tag{3.2.2}$$

Proof:

It is known from the proof of lemma 1 that for a fixed λ_s, the function $(\frac{\partial E_s(\lambda_s, N_s^v)}{\partial N_s^v})$ is strictly decreasing and continuous. So the expression $v(\frac{\partial E_s(\lambda_s, N_s^v)}{\partial N_s^v})$ is continuous and strictly decreasing in N_s^v, when v is a positive number. For a constant c_s, the larger the value v, the smaller the expression $(\frac{\partial E_s(\lambda_s, N_s^v)}{\partial N_s^v})$ will be required to satisfy the equality condition of expression (3.2.2). This requires a larger N_s^v value. As a result if $v^1 > v^2 > 0$ and if $N_s^{v^1}$ and $N_s^{v^2}$ both exist then $N_s^{v^1} > N_s^{v^2}$. □

With Lemma 2, we shall now show that MP provides solutions with *strictly* better GoS guarantees than MC in theorem 2.

Theorem 2

If (3.2.1) holds, moreover, if the routing scheme is monotonic in the allocated capacities, and does not reduce the importance of a link as its capacity increases. Then the solution of the MP formulation will deliver strictly better GoS guarantees than that of the MC formulation. The optimal solution for (2.1.1) will be the same as its unconstrained counterpart.

Proof :

Consider the case that the optimal solution $(\Lambda_s^{c*}, N^{c*}(\Lambda_s^{c*}))$ of the MC formulation is regarded as the initial solution of the MP formulation. Assume that the optimization process attempt to solve (3.1.7) at each iteration and this process converges to the optimality of MP [3]. Assume also that the initial vector [u_{ij}] to be $\mathbf{0}^T$, this vector will increase iff some GoS constraint(s) is/are violated. Substitute $(\Lambda_s^{c*}, N^{c*}(\Lambda_s^{c*}))$ into the optimality condition of MP (3.1.7). Recall that ((3.2.1) holds, by lemma 2 the allocated capacities on all the link strictly increases at the end of the first MP iteration. Since it is assumed that the importance of a link does not decrease as its capacity increases, therefore $f_{2,s}^{ij}$ is not decreased in the second MP iteration. As a result $\sum_{ij \in \theta_s} w^{ij} \lambda^{ij} f_{2,s}^{ij}(.)$ is non-decreasing in the second MP iteration, by lemma 2 the allocated capacities are non-decreasing in the second iteration. This augmentation of capacities continues until the stationary point at N^{p*} is reached. So we have $N^{p*} > N^{c*}$, where N^{p*} is the capacity vector at the optimality of the MP. Now because of the monotonic assumption of the routing scheme and also because $N^{p*} > N^{c*}$, the GoS guarantees offered by N^{p*} is strictly better than that being offered by N^{c*}. Moreover, since the optimization process augments the capacities, so $[u_{ij}]=\mathbf{0}^{\mathrm{T}}$ for all the iterations and we have $[u_{ij}^{*}]=\mathbf{0}^{\mathrm{T}}$. Because $[u_{ij}^{*}]=\mathbf{0}^{\mathrm{T}}$, so the solution of MP remains the same even if all the GoS constraints are relaxed. □

By definition the MP formulation is to maximize the profit so the profit generated by its solutions must not be worse than that of the MC. We shall show in the following theorem that MP provides solution with *strictly* more profit than the MC formulation if condition (3.2.1) is satisfied.

Theorem 3

If the routing scheme is monotonic in the allocated capacities and if condition (3.2.1) holds, the MP formulation delivers solution with *strictly* more profit than the MC formulation.

Proof:

Note that from the proof of theorem 2, when condition (3.2.1) holds, u_{ij}^{*} in (3.1.7) are all zero at optimality, this is due to the theorem 2 and the complementary slackness condition. Now expression (3.1.7) can be written as (3.2.3):

$$c_s = [\sum_{ij\in\theta_s} \lambda^{ij} w^{ij} \times f_{2,s}^{ij}(\lambda_{s'}^{p*}, N_{s'})](1-h_s)\frac{-\partial E_s(\lambda_s, N_s)}{\partial N_s}) \qquad (3.2.3)$$

To show the profit generated by the solution of MP formulation is strictly larger if condition (3.2.1) holds, we first rewrite the R.H.S of (3.2.1) as

$$\sum_{ij\in\theta_s} w^{ij}\lambda^{ij} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*}) = \sum_{ij\in\theta_s} v_{ij}^{*} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*}) + x_s^{*}\sum_{ij\in\theta_s} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*}) \quad ,$$

where $x_s^{*} > 0$ for all s. We substitute the optimal solution $(\Lambda_s^{c*}, N^{c*}(\Lambda_s^{c*}))$ of MC with this expression into the RHS of (3.2.3). Rearranging the terms we have (3.2.4):

$$\sum_{ij\in\theta_s} v_{ij}^{*} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*})(1-h_s)\frac{-\partial E_s(\lambda_s^{c*})}{\partial N_s^{c*}} + x_s^{*}\sum_{ij\in\theta_s} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*})(1-h_s)\frac{-\partial E_s(\lambda_s^{c*})}{\partial N_s^{c*}} \qquad (3.2.4)$$

Expression (3.2.4) denotes the marginal profit from the link by allocating a capacity of N_s^{c*} when (3.2.1) holds. By referring to (3.1.8), the first term in (3.2.4) equals to c_s, so (3.2.4) can be reduced to (3.2.5).

$$c_s + x_s^{*}[\sum_{ij\in\theta_s} f_{2,s}^{ij}(\lambda_{s'}^{c*}, N_{s'}^{c*})](1-h_s)(\frac{-\partial E_s(\lambda_s^{c*})}{\partial N_s^{c*}}) \qquad (3.2.5)$$

We know that $\frac{-\partial E_s(\lambda_s^{c*})}{\partial N_s^{c*}}$, $f_{2,s}^{ij}$ are positive, and h_s is a probability, therefore the second term of (3.2.5) is positive. As a result, from expression (3.2.5) one can conclude that at the optimality of MC formulation (i.e. N_s^{c*}), the marginal reward is strictly larger than marginal cost c_s. This implies strictly more profit can be generated if extra capacities are allocated, and the profit attend maximum at the solution of MP (i.e. (3.2.3)) □

From the analysis in theorem 3, we can gain the insight that the solutions of MC offer strictly worse profit than that of MP because the MC formulation merely allocates the minimum amount of capacities to satisfy the GoS requirements. Additional profits that could have been gained are lost. The main conclusion from sub-Sections 3.1 and 3.2 is that the MP formulation offers no worse performance than the MC formulation in all scenarios. When rewards are low and the cost is the major concern, the MP formulation minimizes the cost just like MC. When the rewards are high enough, the MP formulation switches to maximize the profit. In this sense, designing the SON network using the MP formulation is superior to the MC formulation.

4 AN ILLUSTRATIVE EXAMPLE

We consider a simple SON network as shown in figure 2 for illustration. In this example, SON Gateway A is required to send a Poisson stream of 12 connections per unit time, on the behalf of its clients, to Gateway B. Gateway C is required to send a Poisson stream of 24 connections per unit time, on its clients' behalf, to gateway B. We consider *two* typical routing schemes. Namely (*I*) the direct-link scheme that uses only the direct links, and (*II*) the minimum cost scheme that uses the cheapest path. Without loss of generality we assume the mean holding times of the connections to be identically distributed with unit mean. The costs of leasing one unit of bandwidth of capacity for one unit of time are 1 unit, 2 units and 50 units respective for links *1*, *2*, and *3*. The allocated capacities are required to be integral values. Assume the hardware failure probabilities for the links are $h_1 = h_2 = h_3 = 0.01$. Let the GoS/reliability requirement for the two streams of connections be 0.9 and 0.95 respectively (i.e. at most 0.1/0.05 probability of failure on an average). The multiplier vectors for the direct-link scheme and the minimum cost scheme are found to be (1994.9, 202.1) and (62.7, 180.6), which translates to service charge vectors of (166.24, 8.42) and (5.22, 7.53). We set the service charges to (166, 8) and (5, 7) respectively for the routing schemes *I* and *II*. Thus condition (3.1.4) is satisfied. Note in tables 1 and 2, for both routing schemes, MP and MC formulations give the same solution. Which is exactly the result being depicted in theorem 1.

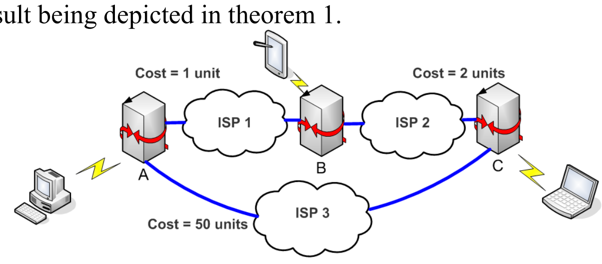

Figure 2. A simple SON network.

We then set the service-charge vectors to (167,9) and (6,8). It is easy to verify that (3.2.1) is satisfied. Tables 1 and 2 again summarize the results obtained. Since the solutions are rounded up to integers from the real-valued optimal solutions, the GoS constraints may not be tight even for the MC formulation. But note that in the tables when the service charges satisfy (3.2.1), the solution of the MP formulation indeed offers strictly lower blocking probabilities for the both routing schemes, which is the result to be expected as indicated by theorem 2. The profits of MP designs also surpass that of the MC designs for the two routing schemes, and this illustrates the results proven in theorem 3.

Table 1. Capacity allocation results for routing scheme I

Direct Link Routing	**MP Service charge (167, 9) units**	**MP Service charge (166, 8) units**	**MC**
Reliability of the design	**(0.93, 0.96)**	**(0.91, 0.95)**	**(0.91, 0.95)**
Allocated Capacities	**(0,31,16)**	**(0,30,15)**	**(0,30,15)**
Cost	**862**	**810**	**810**
Objective value	**-1209.5**	**-1175.5**	**810**
Expected Profit	**1209.5**	**1175.5**	**1209.1/1175.5**

Table 2. Capacity allocation results for routing scheme II

Direct Link Routing	**MP Service charge (6, 8) units**	**MP Service charge (5, 7) units**	**MC**
Reliability of the design	**(0.91, 0.96)**	**(0.90, 0.95)**	**(0.90, 0.95)**
Allocated Capacities	**(17,43,0)**	**(17,42,0)**	**(17,42,0)**
Cost	**103**	**101**	**101**
Objective value	**-146.8**	**-113.0**	**101**
Expected Profit	**146.8**	**113.0**	**146.7/113.0**

5 Conclusions and Future Works

We studied a class of Service Overlay Network (SON) capacity allocation problem with GoS/reliability considerations. The problem can be formulated as Maximum Profit (MP) or Minimum Cost (MC) optimization problems. We showed the condition such that the MP and MC approaches offer the same solutions. Intuitively we can view the region that the MP and MC are equivalent to be the region that is not profitable to offer the required GoS guarantees. In this region the MP approach minimizes the cost in an exactly same way as the MC approach. While in the profitable regions, MP takes advantage of that and allocates more resource whenever necessary to avoid potential profit losses. This is an attractive feature because the MP approach automatically "decides" whether to minimize the cost or to maximize the profit in an optimal sense. While the MC approach, on the other hand, does not have this capability. In cases that GoS requirements are based on some non-economic considerations (for example GoS could be decided by external entities like the regulating authorities), the MP approach will provide solutions with better value than that of the MC approach as the MC approach will be sticking blindly to the potential arbitrary GoS requirements. The computation complexities for both the MP and the MC will be similar in a sense that major efforts for both approaches will be on solving the optimal routing problem and calculating the gradients of B^{ij}. Therefore the MP approach appears to be a better option for SON network designs. Owning to the space limitation, the results in this paper do not cover the regions such that the vector $\boldsymbol{B}$ is positive but the condition (3.1.4) is not satisfied, and regions such that condition (3.2.1) is partially satisfied (i.e. it is satisfied on some of the links, but not on the other links). The results for these regions will be included in a more comprehensive extension of the current paper. From the results in section 3.2, one can see that the MC Lagrangian multipliers act as a form of threshold for pricing under the GoS constraints. They can be employed as a form reference metric for the service charges. This result will also be discussed in an extension of the current paper.

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